Boundary layer formation in the transition from the Porous Media Equation to a Hele-Shaw flow

Let $u_m(x, t)$ be the solution to the Porous Media Equation, $u_t = \Delta u^m$, in a domain $\Omega \subset \mathbb{R}^n$, with initial data $u_m(x, 0) = f(x)$ and boundary data $u^m_m(x, t) = g(x)$. Let $v_m \equiv u^m_m$. We prove the convergence as $m$ goes to infinity of the pair $(u_m, v_m)$ to a pair $(u_\infty, v_\infty)$ which is a weak solution of the Hele-Shaw problem with boundary data $v_\infty = g$ and initial data $u_\infty(x, 0) = f(x)$, where $f(x)$ is the projection of the initial data $f(x)$ into a ‘mesa’. We also prove the convergence of the positivity sets of the functions $u_m$ to the positivity set of $u_\infty$. For large but finite $m$ a boundary layer connecting the initial data $f(x)$ and its projection $\tilde{f}(x)$ appears. We analyze the convergence of solutions and positivity sets in this boundary layer by introducing a suitable time scale. All our results hold true also for the Cauchy problem ($\Omega = \mathbb{R}^n$, no boundary data).