NUMERICAL SIMULATION OF TWO-DIMENSIONAL BINGHAM FLUID FLOW BY SEMISMooth NEWTON METHODS∗

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Abstract. This paper is devoted to the numerical simulation of two-dimensional stationary Bingham fluid flow by semi-smooth Newton methods. We analyze the modeling variational inequality of the second kind, considering both Dirichlet and stress-free boundary conditions. A family of Tikhonov regularized problems is proposed and the convergence of the regularized solutions to the original one is verified. By using Fenchel’s duality, optimality systems which characterize the original and regularized solutions are obtained. The regularized optimality systems are discretized using a finite element method with (cross-grid P1)-Q0 elements for the velocity and pressure, respectively. A semismooth Newton algorithm is proposed in order to solve the discretized optimality systems. Using an additional relaxation, a descent direction is constructed from each semismooth Newton iteration. Local superlinear convergence of the method is also proved. Finally, we perform numerical experiments in order to investigate the behavior and efficiency of the method.

1. Introduction

Bingham fluids are visco-plastic materials which behave like incompressible fluids in the regions where the stress is larger than a given yield and like solids in the regions where the stress remains below that threshold. Examples of these kind of materials include toothpaste, concentrated mineral suspensions, slurries, lava, among others. The mathematical models for such materials involve the constituent law for viscous incompressible fluids with an extra stress tensor component modeling the visco-plastic effects.

The analysis of the Bingham fluid flow variational inequality was carried out in [11], where the authors investigated existence, uniqueness and regularity of the solution for the steady and instationary flows in a reservoir. Existence and extra regularity results for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions are also studied in [13, 14, 15]. We also refer to [16] for a rather complete theoretical treatment.

The numerical solution of the stationary Bingham fluid flow problem is studied in e.g. [20, 18, 10, 9] for flows in cylindrical pipes and in [33, 36, 22] for flows in two-dimensional geometries. In [36], the author investigates the application of an augmented Lagrangian method together with an incompressible finite element approximation for the flow in a driven cavity. In [33, 22], the two-dimensional flow problem is studied by using a C∞-regularization of the constitutive equations and then discretizing the resulting system of PDEs. In addition, in [22] the authors propose a preconditioned iterative scheme for such regularized systems.

An alternative approach for the numerical solution of the stationary problem consists in solving the time dependent problem until a steady state is reached. The instationary Bingham flow problem in a reservoir is numerically studied in [20], where an external finite element approximation and an Uzawa-type algorithm are utilized. Algorithms based on operator splitting techniques are studied in [10, 19, 38] for the Bingham flow in reservoirs and driven cavities. In [39] the application of an augmented Lagrangian method for the simulation of the instationary Bingham flow in a more general two-dimensional geometry is considered.

In this paper, we are concerned with Bingham fluid flow in a given domain Ω ⊂ R2, considering non-homogeneous Dirichlet and stress-free boundary conditions. We analyze the modeling elliptic variational inequality of the second kind as an equivalent minimization problem and, using Fenchel’s

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duality, we obtain an optimality system which characterizes the primal and dual solutions. Since the solution to the dual problem is not unique, a family of Tikhonov regularized problems is introduced and the convergence of the regularized solutions to the original one is studied. For a similar treatment concerning Bingham fluid flow in cylindrical pipes we refer to [9].

For the discretization of each regularized optimality system, a finite element method with (cross-grid $P_1 - Q_0$) elements is utilized. The chosen pair is known to satisfy the Ladyzhenskaya - Babuška - Brezzi condition (cf. [6, Sec. 12.5]) and allows a direct relation between the discrete primal and dual variables to be obtained.

For the solution of the resulting system of nonsmooth equations, we propose a semismooth Newton algorithm. After an additional relaxation of the incompressibility condition (see [23, pg. 125]), a modified reduced system matrix is constructed, which leads to a descent direction in each semismooth Newton iteration. The local superlinear convergence of the method is also proved.

The paper is organized as follows. In Section 2, we state the problem and summarize some existence and uniqueness results. In Section 3, by using Fenchel’s duality theory, we derive a necessary condition for our problem and characterize the primal and dual solutions by an optimality system of equations. Since the optimality system for the original problem is ill-posed, in Section 4 we introduce a family of Tikhonov regularized problems and prove the convergence of the regularized solutions to the original one. In Section 5, we analyze the discretization of the regularized optimality systems by using a (cross-grid $P_1 - Q_0$) finite element approximation. In section 6, we construct and analyze the semismooth Newton algorithm. In particular, we prove the local superlinear convergence rate of the algorithm. In section 7, several numerical experiments are carried out in order to verify the theoretical properties of the algorithm.

2. PROBLEM STATEMENT

We start by introducing some notation. The Euclidean inner product in $\mathbb{R}^d$ and its associated norm are denoted $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. The Frobenius scalar product in $\mathbb{R}^{d \times d}$ and its associated norm are defined by

$$
(A : B) := \text{tr}(AB^T) \quad \text{and} \quad \|A\| := \sqrt{(A : A)}, \quad \text{for} \quad A, B \in \mathbb{R}^{d \times d}, \quad d \geq 2,
$$

where $\text{tr}$ stands for the trace of the matrix.

We use the notation $\langle \cdot, \cdot \rangle_{X^*, X}$ for the dual pairing between a Banach space $X$ and its corresponding dual space $X^*$. The scalar product in a Hilbert space is denoted by $\langle \cdot, \cdot \rangle_X$ and its associated norm by $\|\cdot\|_X$. Throughout, $L^2(\Omega)$ stands for the space $(L^2(\Omega))^d$ and $H^1(\Omega)$ for $(H^1(\Omega))^d$. $L^{2 \times 2}(\Omega)$ stands for the space of $(d \times d)$-matrices of $L^2(\Omega)$-functions. We endow this space with the norm $\|p\|_{L^{2 \times 2}}$, which is induced by the following scalar product

$$
(p, q)_{L^{2 \times 2}} := \int_{\Omega} (p(x) : q(x)) \, dx.
$$

With this scalar product, $L^{2 \times 2}(\Omega)$ is a Hilbert space, since $L^{2 \times 2}(\Omega)$ endowed with the scalar product (2.1) is isomorph to the space $(L^2(\Omega))^{d \times d}$ endowed with the usual $L^2(\Omega)$-scalar product in product spaces (see Section 5.1).

Throughout the paper, the notation $\text{div} \, y$ stands for the divergence of a vector field $y : \Omega \rightarrow \mathbb{R}^d$. Further, the notation $\text{Div} \, q$ stands for the rowwise divergence operator of a matrix $q : \Omega \rightarrow \mathbb{R}^{d \times d}$.

Let $\Omega$ be an open bounded set of $\mathbb{R}^d$, $d \in \{2, 3\}$, with Lipschitz boundary $\Gamma$. We assume that $\Gamma = \Gamma_D \cup \Gamma_0 \cup \Gamma_N$, where $\Gamma_D \cap \Gamma_0$ has non-null measure and $\Gamma_N$ may have null measure. In this paper, we are concerned with the following boundary value-problem: find a velocity field $y : \Omega \rightarrow \mathbb{R}^d$
and a scalar field $p: \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
\text{Div} \, \sigma - \nabla p + f &= 0 & \text{in } \Omega, \\
\text{div} \, y &= 0 & \text{in } \Omega, \\
\sigma_{\text{Tot}} &= -p \cdot I + \sigma, \\
\sigma &= 2\mu \mathcal{E}y + \sqrt{2} g \frac{\mathcal{E}y}{\|\mathcal{E}y\|} & \text{if } \mathcal{E}y \neq 0, \\
\|\sigma\| &\leq g & \text{if } \mathcal{E}y = 0, \\
\sigma_{\text{Tot}} \cdot \mathbf{n} &= 0 & \text{on } \Gamma_N, \\
y &= y_D & \text{on } \Gamma_D, \\
y &= 0 & \text{on } \Gamma_0,
\end{aligned}
$$

(2.2)

where $\mu$ stands for the viscosity coefficient, $g$ for the plasticity threshold (yield stress) and $f$ is the volume density of given forces. The total Cauchy tensor is denoted by $\sigma_{\text{Tot}}$ and $\mathcal{E}$ stands for the deformation or rate of strain tensor, whose components are given by

$$
(\mathcal{E}y)_{ij} := \frac{1}{2} \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right), \quad \text{for } y = (y_1, \ldots, y_d)^\top.
$$

System (2.2) corresponds to the strong formulation of the stationary Bingham fluid flow model.

Introducing the convex set $Y_D \subset \mathbf{H}^1(\Omega)$ by

$$
Y_D := \{ v \in \mathbf{H}^1(\Omega) : \text{div} \, v = 0 \text{ in } \Omega, \, v = 0 \text{ on } \Gamma_0 \text{ and } v = y_D \text{ on } \Gamma_D \}
$$

and following [11, 13, 37], a weak form of problem (2.2) is given by the following variational deformation or rate of strain tensor, whose components are given by the necessary condition of the following minimization problem

$$
\inf_{y \in Y_D} J(y) := \frac{1}{2} a(y, y) + \tilde{g}j(y) - (f, y)_{L^2},
$$

Existence and uniqueness of solutions for (P) immediately follows from the coercivity of the bilinear form $a(\cdot, \cdot)$ and the convexity of $j(\cdot)$ (See [11] and [31, Th. 1.6]).

3. The Fenchel Dual

In this section, we study the dual problem of (P) by using Fenchel’s duality theory. We start the analysis with some definitions. Let $Y$ be a space of divergence free velocity fields, given by

$$
Y := \{ y \in \mathbf{H}^1(\Omega) : \text{div} \, y = 0 \text{ and } y = 0 \text{ on } \Gamma_0 \}.
$$

$K$ stands for the subspace of $\mathbf{L}^{2\times 2}(\Omega)$ of symmetric matrices, i.e.,

$$
K := \{ p \in \mathbf{L}^{2\times 2}(\Omega) : p_{ij} = p_{ji} \text{ a.e. in } \Omega, \text{ for all } i, j = 1, \ldots, d \},
$$

and the operator $\Lambda \in \mathcal{L}(Y, K)$ is defined by $\Lambda v := \mathcal{E}v$.

Next, let $\mathcal{F}: Y \to \mathbb{R}$ be defined by

$$
\mathcal{F}(y) := \begin{cases} 
\frac{1}{2} a(y, y) - (f, y)_{L^2} & \text{if } y \in Y_D \\
+\infty & \text{otherwise},
\end{cases}
$$

and $\mathcal{G}: K \to \mathbb{R}$ be defined by $\mathcal{G}(q) := \tilde{g} \int_\Omega \|q\| \, dx.$
Using these definitions, we can rewrite problem \((\mathcal{P})\) in the form
\[
\inf_{y \in Y} \{ \mathcal{F}(y) + \mathcal{G}(\Lambda y) \}.
\]
Following [12, pp. 60-61], we know that the dual problem is given by
\[
\sup_{q \in K^{*}} \{ -\mathcal{F}^{*}(-\Lambda^{*}q) - \mathcal{G}^{*}(q) \},
\]
where \(\mathcal{F}^{*} : Y^{*} \to \mathbb{R}\) and \(\mathcal{G}^{*} : K^{*} \to \mathbb{R}\) denote the convex conjugate functionals of \(\mathcal{F}\) and \(\mathcal{G}\) respectively, and \(\Lambda^{*} \in L(K^{*}, Y^{*})\) is the adjoint operator of \(\Lambda\).

Now, we calculate the convex conjugate functionals \(\mathcal{F}^{*}\) and \(\mathcal{G}^{*}\). Let \(q \in K^{*}\) and let us identify \(K\) with its topological dual \(K^{*}\). Thus, we have that
\[
\mathcal{F}^{*}(-\Lambda^{*}q) = \sup_{y \in Y} \{ \langle -\Lambda^{*}q, y \rangle_{Y^{*}} - \mathcal{F}(y) \}
\]
(3.3)
\[
= \sup_{y \in Y_{D}} \left\{ -\langle q, \mathcal{E}y \rangle_{L^{2} \times L^{2}} - \frac{1}{2} a(y, y) + (f, y)_{L^{2}} \right\}.
\]

From [31, Th. 1.2, p. 9], we conclude that the solution \(y_{q} \in Y_{D}\) of (3.3) satisfies the following variational inequality
\[
a(y_{q}, v - y_{q}) + \langle q, \mathcal{E}(v - y_{q}) \rangle_{L^{2} \times L^{2}} \geq (f, v - y_{q})_{L^{2}}, \text{ for all } v \in Y_{D},
\]
which is equivalent to
\[
a(y_{q}, z) + \langle q, \mathcal{E}z \rangle_{L^{2} \times L^{2}} - (f, z)_{L^{2}} = 0, \text{ for all } z \in Y_{0,D},
\]
where the space \(Y_{0,D}\) is defined by
\[
Y_{0,D} := \{ y \in H^{1}(\Omega) : \text{div } y = 0 \text{ and } y = 0 \text{ on } \Gamma_{D} \cup \Gamma_{0} \}.
\]

Thus, we conclude that
\[
\mathcal{F}^{*}(-\Lambda^{*}q) = -\frac{1}{2} a(y_{q}, y_{q}) - \langle q, \mathcal{E}y_{q} \rangle_{L^{2} \times L^{2}} + (f, y_{q})_{L^{2}}.
\]

Note that \(y_{q}\) is just an auxiliary variable which depends on \(q\).

**Lemma 3.1.** The inequality
\[
(q, p)_{L^{2} \times L^{2}} \leq \tilde{g} \int_{\Omega} \| p(x) \| \, dx, \text{ for all } p \in K
\]
is equivalent to
\[
\| q(x) \| \leq \tilde{g} \text{ a.e. in } \Omega.
\]

**Proof.** Assume that (3.8) does not hold, i.e., assume that \(S := \{ x \in \Omega : \tilde{g} - \| q(x) \| < 0 \} \) has positive measure. Choosing \(\hat{p} \in K\) such that
\[
\hat{p}(x) := \begin{cases} 
q(x) & \text{in } S \\
0 & \text{in } \Omega \setminus S
\end{cases}
\]
leads to
\[
\tilde{g} \int_{\Omega} \| \hat{p}(x) \| \, dx - \int_{\Omega} (q : \hat{p}) \, dx = \int_{S} (\tilde{g} - \| q(x) \|) \| q(x) \| \, dx < 0,
\]
which is a contradiction to (3.7). Thus, (3.7) implies (3.8). Reciprocally, since \(\| q(x) \| \leq \tilde{g}\), a.e. in \(\Omega\) and thanks to the Cauchy-Schwarz inequality, we obtain, for an arbitrary \(p \in K\), that
\[
\tilde{g} \int_{\Omega} \| p(x) \| \, dx - \int_{\Omega} (q : p) \, dx \geq \int_{\Omega} (\tilde{g} - \| q(x) \|) \| p(x) \| \, dx \geq 0.
\]
\[\square\]

Lemma 3.1 immediately implies that
\[
\mathcal{G}^{*}(q) = \begin{cases} 
0 & \text{if } \| q(x) \| \leq \tilde{g}, \text{ a.e. in } \Omega \\
+\infty & \text{otherwise}
\end{cases}
\]
(3.9)
Finally, by plugging (3.6) and (3.9) in (3.2), we can specify the dual problem as

\[
(P^*) \quad \begin{cases} 
\sup_{|q| \leq \tilde{\beta}} J^*(q) := \frac{1}{2} a(y_q, y_q) + (q, \mathcal{E} y_q)_{L^2} - (f, y_q)_{L^2}.
\end{cases}
\]

where \( y_q \) satisfies

\[
a(y_q, z) + (q, \mathcal{E} z)_{L^2} = (f, z)_{L^2}, \text{ for all } z \in Y_{0,D}.
\]

We now prove that the dual problem \((P^*)\) has at least one solution and that no duality gap occurs, i.e.,

\[
(3.10) \quad \inf_{y \in Y} \{ F(y) + G(Ay) \} = \sup_{q \in K^*} \{-F^* (-\Lambda^* q) - G^* (q)\}.
\]

First, it is easy to see that \( G \) is a convex and continuous functional, whose domain is the whole space \( K \). Thus, it is lower semicontinuous. Next, we prove that \( F \) is also lower semicontinuous. Let

\[
\text{level}_\alpha F := \{ v \in Y : F(v) \leq \alpha \}
\]

be the level set of height \( \alpha \in \mathbb{R} \) of \( F \). First, note that \( \text{level}_0 F \subset Y_D \). Since \( F \) is continuous and convex and since \( Y_D \) is closed, the set \( \text{level}_0 F \) is also convex and closed. Therefore, [7, Th. 2.1, p. 28] and [29, Lem. 2.11] imply that \( F \) is a lower semicontinuous functional in \( Y \). Moreover, we have that both \( F \) and \( G \) are proper and that there exists at least one \( y_0 \in Y_D \) with \( F(y_0) < \infty \) and \( G(Ay_0) < \infty \) and \( G \) is continuous at \( Ay_0 \). Thus, [12, Th. 4.1, p. 59] and [12, Rem. 4.2, p. 60] imply (3.10) and the existence of at least one solution \( \tilde{q} \in K \).

### 3.1. Extremality conditions

From Fenchel’s duality theory it follows that the solutions \( \overline{y} \) and \( \overline{q} \) are equivalently characterized by the following extremality conditions:

\[
\begin{align*}
(3.11) & \quad -\Lambda^* \overline{q} \in \partial F(\overline{y}) \\
(3.12) & \quad \overline{q} \in \partial G(\mathcal{E}\overline{y}).
\end{align*}
\]

Let us start by studying (3.11). From the definition of the subdifferential, we have that

\[
\forall \alpha \in (0,1), \quad \mathcal{F}(w) - \mathcal{F}(\overline{y}) \geq - (\overline{q}, \mathcal{E}(w - \overline{y}))_{L^2} \quad \text{for all } w \in Y.
\]

In particular, (3.13) holds for \( w = \overline{y} + \alpha (v - \overline{y}) \), with \( v \in Y_D \) and \( \alpha \in (0,1) \). Since \( Y_D \) is a convex set, we have that \( w \in Y_D \). Moreover, we have that the following expression is well defined

\[
\frac{\mathcal{F}(\overline{y} + \alpha (v - \overline{y})) - \mathcal{F}(\overline{y})}{\alpha} \geq - (\overline{q}, \mathcal{E}(v - \overline{y}))_{L^2} \quad \text{for all } v \in Y_D, \alpha \in (0,1).
\]

Therefore, by taking limits as \( \alpha \to 0 \), we obtain, since \( \mathcal{F} \) is Gâteaux differentiable at \( \overline{y} \in Y_D \), that

\[
a(\overline{y}, v - \overline{y}) + (\overline{q}, \mathcal{E}(v - \overline{y}))_{L^2} \geq (f, v - \overline{y})_{L^2}, \quad \text{for all } v \in Y_D,
\]

which is equivalent to

\[
(3.15) \quad a(\overline{y}, z) + (\overline{q}, \mathcal{E}z)_{L^2} - (f, z)_{L^2} = 0, \quad \text{for all } z \in Y_{0,D}.
\]

Next, we analyze (3.12). From the definition of the subdifferential it follows that

\[
\tilde{g} \left( \int_{\Omega} \| E y \| dx - \int_{\Omega} \| p \| dx \right) \leq (q, \mathcal{E} y - p)_{L^2} \quad \text{for all } p \in K.
\]

Then, for \( p = 0 \), we obtain that

\[
\tilde{g} \int_{\Omega} \| E y \| dx \leq (q, \mathcal{E} y)_{L^2},
\]

which implies, since \( |\overline{q}| \leq \tilde{g} \), a.e. in \( \Omega \), that

\[
(3.16) \quad \tilde{g} \int_{\Omega} \| E y \| dx = (\overline{q}, \mathcal{E} y)_{L^2}.
\]

Furthermore, (3.8) and the Cauchy-Schwarz inequality, applied in the Frobenius scalar product, imply that

\[
\tilde{g} \| \mathcal{E} y(x) \| - (\mathcal{E} y(x)) \geq 0, \quad \text{a.e. in } \Omega,
\]

which, together with (3.16), yields that

\[
(3.17) \quad \tilde{g} \| \mathcal{E} y(x) \| = (\overline{q}(x), \mathcal{E} y(x)), \quad \text{a.e. in } \Omega.
\]
Definition 3.1. We define the active and inactive sets for \((S)\) by
\[
\mathcal{A} := \{ x \in \Omega : \| \nabla \mathbf{y}(x) \| \neq 0 \} \quad \text{and} \quad \mathcal{J} := \Omega \setminus \mathcal{A}, \quad \text{respectively.}
\]

Let us point out that (3.8), the expression (3.17) and the Cauchy-Schwarz inequality imply that \(\| \mathbf{v}(x) \| = \tilde{g} \) a.e. in \(\mathcal{A}\).

Summarizing, the optimality system for \((P)\) and \((P^*)\) is given by the following system
\[
\begin{cases}
\mathbf{y} \in Y_D \\
a(\mathbf{y}, \mathbf{z}) + \mathbf{E}(\mathbf{z})_{L^2} = (f, \mathbf{z})_{L^2}, \quad \text{for all } \mathbf{z} \in Y_{0,D}, \\
\| \mathbf{v}(x) \| \leq \tilde{g}, \quad \text{a.e. in } \Omega, \\
\tilde{g} |\mathbf{v}(x)\| = (\mathbf{v}(x) : \nabla \mathbf{y}(x)), \quad \text{a.e. in } \Omega.
\end{cases}
\]

Since system \((S)\) involves the rowwise divergence operator, the multiplier \(q\) is not unique.

4. Regularization

To overcome the difficulties related to the non-uniqueness of the solution to system \((S)\), we propose a Tikhonov-type regularization of \((P^*)\). This procedure leads also to a local regularization of the non-differentiable term \(j(v)\) in \((P)\). This penalization-type smoothing is motivated by augmented Lagrangian theory, results from optimal control problems and TV-image restoration (see [24, 25, 26, 27, 28]), and was previously considered for steady Bingham flow problems in cylindrical pipes (cf. [9]).

For a parameter \(\gamma > 0\) we consider the following penalized dual problem:
\[
\begin{cases}
\sup_{\|q\| \leq \beta} J_\gamma(q) := \frac{1}{2} a(y_q, y_q) + (q, \mathbf{E} y_q)_{L^2} - (f, y_q)_{L^2} - \frac{1}{\gamma} \| q \|_{L^2}^2 \\
\text{where } y_q \text{ satisfies} \\
a(y_q, \mathbf{z}) + (q, \mathbf{E} \mathbf{z})_{L^2} = (f, \mathbf{z})_{L^2}, \quad \text{for all } \mathbf{z} \in Y_{0,D}.
\end{cases}
\]

The regularized problem is obtained from \((P^*)\) by subtracting \(\frac{1}{\gamma} \| q \|_{L^2}^2\) to the objective functional. Further, it is possible to show that this penalization also regularizes the primal problem. In fact, consider the following function \(\Psi_\gamma : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}\), defined by
\[
\Psi_\gamma(A) := \begin{cases}
\tilde{g} \| A \| - \frac{\tilde{g}^2}{2\gamma} & \text{if } \| A \| \geq \frac{\tilde{g}}{2} \\
\frac{\tilde{g}^2}{2} & \| A \| \leq \frac{\tilde{g}}{2}.
\end{cases}
\]

From ([1, Th. 2.3]), we conclude that \(\Psi_\gamma\) is a continuously differentiable function, for each \(\gamma > 0\). Therefore, by using this function, which is a regularization of the Frobenius norm, we can define the following regularized version of \((P)\)
\[
\begin{align*}
\min_{y \in Y_D} J_\gamma(y) := & \frac{1}{2} a(y, y) + \int_\Omega \Psi_\gamma(\mathbf{E} y) \, dx - (f, y)_{L^2}.
\end{align*}
\]

By replacing the functional \(G\) in (3.1) by
\[
G_\gamma(p) = \int_\Omega \Psi_\gamma(p(x)) \, dx,
\]
where \(\Psi_\gamma\) is given by (4.1), it is possible to prove (see [21, pg. 106]) that problem \((P^*_\gamma)\) is the dual problem of \((P_\gamma)\) and that there is not a duality gap, i.e.,
\[
J_\gamma(q_\gamma) = J_\gamma(y_\gamma),
\]

where \(q_\gamma\) and \(y_\gamma\) denote the solutions to \((P^*_\gamma)\) and \((P_\gamma)\) respectively.

Remark 4.1. The regularization procedure turns the objective functional \(J^*\) into the functional \(J_\gamma^*\), which is a \(K\)-uniformly concave functional, defined in a convex set of \(H^1(\Omega)\). Therefore, problem \((P^*_\gamma)\) admits a unique solution \(q_\gamma \in K\) for each fixed \(\gamma > 0\). Furthermore, the coercivity in \(Y\) of the form \(a(\cdot, \cdot)\) and the strict convexity of \(J_\gamma\) imply the existence of a unique solution for \((P_\gamma)\) (see [31, Th. 1.6]).
Next, we characterize the solutions to \((P_\gamma)\) and \((P^*_\gamma)\), \(y_\gamma\) and \(q_\gamma\), respectively. Fenchel’s duality theory implies that
\begin{align*}
(4.4) & \quad -\Lambda^* q_\gamma \in \partial F(y_\gamma), \\
(4.5) & \quad q_\gamma \in \partial G_\gamma(\mathcal{E} y_\gamma).
\end{align*}
Since \(y_\gamma \in Y_D\) and because of the regularization \((4.2)\), \(F\) and \(G_\gamma\) are differentiable in \(y_\gamma\) and \(\mathcal{E} y_\gamma\), respectively. Thus, both \(\partial F(y_\gamma)\) and \(\partial G_\gamma(\mathcal{E} y_\gamma)\) reduce to the respective Gateaux derivatives.

Let us characterize \((4.4)\) and \((4.5)\). Due to the fact that \((4.4)\) is similar to equation \((3.11)\), we conclude that \((4.4)\) yields
\begin{equation}
(4.6) \quad a(y_\gamma, z) + (q_\gamma, \mathcal{E} z)_{L^2\times 2} - (\Gamma, z)_{L^2} = 0, \text{ for all } z \in Y_{0,D}.
\end{equation}
Further, due to the differentiability of \(G_\gamma\), and thanks to [1, Th. 2.3], equation \((4.5)\) can be written as
\begin{equation}
(4.7) \quad (q_\gamma, p)_{L^2\times 2} = \tilde{g} \int_{A_\gamma} \left( \frac{\mathcal{E} y_\gamma}{\|\mathcal{E} y_\gamma\|} : p \right) dx + \gamma \int_\Omega (\mathcal{E} y_\gamma : p) dx, \text{ for } p \in K,
\end{equation}
which implies that
\begin{equation}
(4.8) \quad q_\gamma(x) = \begin{cases} \gamma \mathcal{E} y_\gamma(x) & \text{a.e. in } \Omega \setminus A_\gamma, \\
- \frac{\mathcal{E} y_\gamma(x)}{\|\mathcal{E} y_\gamma(x)\|} & \text{a.e. in } A_\gamma,
\end{cases}
\end{equation}
where \(A_\gamma = \left\{ x \in \Omega : \|\mathcal{E} y_\gamma(x)\| \geq \frac{\tilde{g}}{\gamma} \right\} \text{ a.e.}\). Consequently, the solutions \((y_\gamma, q_\gamma)\) of the regularized problems \((P_\gamma)\) and \((P^*_\gamma)\) satisfy the system
\begin{equation}
(\mathcal{S}_\gamma) \quad \begin{cases} y_\gamma \in Y_D, \\
a(y_\gamma, z) + (q_\gamma, \mathcal{E} z)_{L^2\times 2} = (\Gamma, z)_{L^2}, \text{ for all } z \in Y_{0,D}, \\
q_\gamma(x) = \tilde{g} \max_{\|\mathcal{E} y_\gamma(x)\|} \mathcal{E} y_\gamma(x), \text{ a.e. in } \Omega \setminus A_\gamma,
\end{cases}
\end{equation}
Clearly \(\|q_\gamma(x)\| = \tilde{g} \text{ a.e. in } A_\gamma\), and \(\|q_\gamma(x)\| < \tilde{g} \text{ a.e. in } \Omega \setminus A_\gamma\).

In the following theorem the convergence of the regularized solutions towards the original one is verified.

**Theorem 4.1.** The solutions \(y_\gamma\) of \((P_\gamma)\) converge to the solution \(y\) of \((P)\) strongly in \(H^1(\Omega)\) as \(\gamma \to \infty\). Moreover, the sequence of solutions \(\{q_\gamma\}_{\gamma > 0}\) to \((P^*_\gamma)\) converges (up to a subsequence) to a solution \(q\) to \((P^*)\) weakly in \(L^{2\times 2}(\Omega)\).

**Proof.** Let \((\bar{y}, \bar{q})\) and \((y_\gamma, q_\gamma)\) be solutions to equations \((3.15)\) and \((4.6)\), respectively. By subtracting \((4.6)\) from \((3.15)\), we obtain that
\begin{equation}
(4.9) \quad 2 \mu \int_\Omega (\mathcal{E}(\bar{y} - y_\gamma) : \mathcal{E} z) dx = \int_\Omega (q_\gamma - \bar{q} : \mathcal{E} z) dx, \text{ for all } z \in Y_{0,D}.
\end{equation}
Choosing \(z = \bar{y} - y_\gamma \in Y_{0,D}\), we have that
\begin{equation}
(4.10) \quad 2 \mu \int_\Omega (\mathcal{E}(\bar{y} - y_\gamma) : \mathcal{E}(\bar{y} - y_\gamma)) dx = \int_\Omega (q_\gamma - \bar{q} : \mathcal{E}(\bar{y} - y_\gamma)) dx.
\end{equation}
Next we establish pointwise bounds for \(\{(q_\gamma(x) - \bar{q}(x)) : E(\bar{y} - y_\gamma)(x)\}\) on the following four disjoint sets: \(A \cap A_\gamma\), \(A \cap I_\gamma\), \(A_\gamma \cap I\) and \(I_\gamma \cap I\).

On \(A \cap A_\gamma\): Here, we know that \(\|q_\gamma(x)\| = \|q_\gamma(x)\| = \tilde{g}\) and \(q_\gamma(x) = \tilde{g} \frac{\mathcal{E} y_\gamma(x)}{\|\mathcal{E} y_\gamma(x)\|}\). Thus, due to the Cauchy-Schwarz inequality and \((3.17)\), we have the following pointwise estimate
\begin{equation}
(4.11) \quad ((q_\gamma - \bar{q})(x) : \mathcal{E}(\bar{y} - y_\gamma)(x)) \leq \|q_\gamma(x)\| \|\mathcal{E} y_\gamma(x)\| - \left( \tilde{g} \frac{\mathcal{E} y_\gamma(x)}{\|\mathcal{E} y_\gamma(x)\|} : \mathcal{E} y_\gamma(x) \right)
\end{equation}
On $A \cap I$: Here, we know that $Ey_\gamma(x) = \gamma^{-1}q_\gamma(x)$, $\|q_\gamma(x)\| < \tilde{y}$ and $\|q(x)\| = \tilde{y}$. Hence, from the Cauchy-Schwarz inequality and (3.17), we get
\begin{align}
\langle(q_\gamma - \tilde{q})(x), Ey - y_\gamma(x)\rangle &< \tilde{y}\|Ey(x)\| - \gamma^{-1}\|q_\gamma(x)\|^2 \\
&< \gamma^{-1}(\tilde{y}^2 - \|q_\gamma(x)\|^2) \leq \frac{\tilde{y}^2}{\gamma}.
\end{align}
\label{eq:4.12}

On $A_0 \cap I$: Here, we have that $Ey(x) = 0$, $\|\tilde{q}(x)\| \leq \tilde{y}$ and $q_\gamma(x) = \tilde{y} \frac{Ey_\gamma(x)}{\|Ey_\gamma(x)\|}$. Then, thanks to the Cauchy-Schwarz inequality, we have that
\begin{align}
\langle(q_\gamma - \tilde{q})(x), Ey - y_\gamma(x)\rangle &= \langle(q_\gamma - \tilde{q})(x), Ey_\gamma(x)\rangle \\
&\leq \|\tilde{q}(x)\|\|Ey_\gamma(x)\| - \tilde{y}\|Ey_\gamma(x)\||0.
\end{align}
\label{eq:4.13}

On $I \cap I$: Here, it holds that $Ey(x) = 0$, $Ey_\gamma(x) = \gamma^{-1}q_\gamma(x)$, $\|\tilde{q}(x)\| \leq \tilde{y}$, and $\|q_\gamma(x)\| < \tilde{y}$. Thus, the Cauchy-Schwarz inequality implies that
\begin{align}
\langle(q_\gamma - \tilde{q})(x), Ey - y_\gamma(x)\rangle &\leq \|\tilde{q}(x)\|\|Ey_\gamma(x)\| - \gamma^{-1}\|q_\gamma(x)\|^2 \\
&\leq \gamma^{-1}\tilde{y}\|q_\gamma(x)\| - \gamma^{-1}\|q_\gamma(x)\|^2 \\
&< \gamma^{-1}(\tilde{y}^2 - \|q_\gamma(x)\|^2) \leq \frac{\tilde{y}^2}{\gamma}.
\end{align}
\label{eq:4.14}

From (4.10) and from estimates (4.11)-(4.14), we obtain that
\begin{equation}
\label{eq:4.15}
\frac{1}{\gamma}d(y - y_\gamma, Ey - y_\gamma) < \int_{\Omega} \frac{\tilde{y}^2}{\gamma} dx.
\end{equation}

Next, since $y - y_\gamma \in Y \subset X$ and due to the coercivity of $a(\cdot, \cdot)$ in space $X$ (see Remark 2.1), there exists $C > 0$ such that
\[
a(y - y_\gamma, y - y_\gamma) \geq C \|y - y_\gamma\|_{H^1}^2,
\]
which yields, together with (4.15), that $y_\gamma \rightharpoonup y$ strongly in $H^1(\Omega)$, as $\gamma \to \infty$. Moreover, from (4.9) we obtain that
\[
\lim_{\gamma \to \infty} (q_\gamma - \tilde{q}, Ez)_{L^2 \times 2} = 0, \text{ for all } z \in Y_{0,D},
\]
and, consequently, $J^*(q_\gamma) \rightharpoonup J^*(\tilde{q})$.

On the other hand, since $\|q_\gamma(x)\| \leq \tilde{y}$ a.e. in $\Omega$ for all $\gamma > 0$, there exists some $\hat{q}$ and a subsequence (denoted in the same way) such that
\[
q_\gamma \rightharpoonup \hat{q} \text{ weakly in } L^2 \times 2(\Omega).
\]
The latter together with the strong convergence of $y_\gamma$ towards $y$ in $H^1(\Omega)$ imply that $(\tilde{y}, \hat{q})$ satisfy equation (3.5) and
\[
J^*(q_\gamma) \rightharpoonup J^*(\hat{q}) = J^*(\tilde{q}).
\]

Additionally, since the set $\{\phi \in L^2 \times 2(\Omega) : \|\phi(x)\| \leq \tilde{y} \text{ a.e. in } \Omega\}$ is closed and convex, it is weakly closed and, therefore,
\[
\|\hat{q}(x)\| \leq \tilde{y} \text{ a.e. in } \Omega,
\]
which completes the proof.

\qed

4.1. Recovering the pressure in $(S_\gamma)$. Let $q = (q_{ij})_{i,j=1,...,d} \in K$ and $z = (z_1, \ldots, z_d)^T \in H^1_\partial(\Omega)$. From the definition of the Frobenius scalar product, it follows that
\[
(q, Ez)_{L^2 \times 2} = \sum_{i,j=1}^{d} \int_{\Omega} q_{ij} (Ez)_{ij} dx = \sum_{i=1}^{d} (q_i, \nabla z_i)_{L^2},
\]
where $\mathbf{q}_i = (q_{i1}, \ldots, q_{id}) \in \mathbb{L}^2(\Omega)$, $i = 1, \ldots, d$, stand for the rows of matrix $\mathbf{q}$. Next, since $\text{div} \in \mathcal{L}(\mathbb{L}^2(\Omega), H^{-1}(\Omega))$ is the adjoint operator of $\nabla \in \mathcal{L}(H^1_0(\Omega), \mathbb{L}^2(\Omega))$ (see [20, p. 350]), the following identity holds

$$\sum_{i=1}^d (\mathbf{q}_i, \nabla z_i)_{\mathbb{L}^2} = \sum_{i=1}^d (-\text{div} \mathbf{q}_i, z_i)_{H^{-1}(\Omega), H^1_0(\Omega)},$$

which yields that

$$\begin{align*}
(\mathbf{q}, \mathcal{E}z)_{\mathbb{L}^2} &= (-\text{Div} \mathbf{q}, z)_{H^{-1}(\Omega), H^1_0(\Omega)}, \text{ for all } \mathbf{q} \in \mathbb{K} \text{ and all } z \in H^1_0(\Omega).
\end{align*}$$

Further, let $\mathbf{y} = (y_1, \ldots, y_d)^\top \in Y$. Since $\mathcal{E} \mathbf{y} \in \mathbb{K}$, (4.16) implies that

$$\begin{align*}
(\mathcal{E} \mathbf{y}, \mathcal{E}z)_{\mathbb{L}^2} &= (-\text{Div} (\mathcal{E} \mathbf{y}), z)_{H^{-1}(\Omega), H^1_0(\Omega)}.
\end{align*}$$

Due to the fact that $\text{div} \mathbf{y} = 0$, the following identity holds (see [17, p. 51])

$$\begin{align*}
\sum_{j=1}^d \frac{\partial}{\partial x_j} (\mathcal{E}y)_{ii} &= \frac{1}{2} \sum_{j=1}^d \left( \frac{\partial^2 y_i}{\partial x_j^2} + \frac{\partial^2 y_j}{\partial x_j \partial x_i} \right) = \frac{1}{2} \Delta y_i.
\end{align*}$$

Therefore, from (4.17) and (4.18), we can conclude that

$$\begin{align*}
(\mathcal{E} \mathbf{y}, \mathcal{E}z)_{\mathbb{L}^2} &= \frac{1}{2} (-\Delta \mathbf{y}, \mathbf{z})_{H^{-1}(\Omega), H^1_0(\Omega)}, \text{ for all } \mathbf{y} \in Y \text{ and all } \mathbf{z} \in H^1_0(\Omega).
\end{align*}$$

Finally, let us recall that the solenoidal space $V$ is defined by

$$V := \{ \mathbf{z} \in H^1_0(\Omega) : \text{div} \mathbf{z} = 0 \}.$$

**Proposition 4.2.** Let $\text{meas}(\Gamma_N) > 0$. Then, there exists a unique $p \in \mathbb{L}^2(\Omega)$ such that

$$\begin{align*}
-\mu \Delta \mathbf{y}_\gamma - \nabla \mathbf{q}_\gamma + \nabla p &= \mathbf{f} \text{ in } H^{-1}(\Omega).
\end{align*}$$

**Proof.** First, we recall that $(\mathbf{y}_\gamma, \mathbf{q}_\gamma) \in Y_D \times \mathbb{K}$ satisfy equation (4.6). Moreover, since $V \subset Y_{0,D}$, we have that

$$\begin{align*}
a(\mathbf{y}_\gamma, \mathbf{z}) + (\mathbf{q}_\gamma, \mathcal{E} \mathbf{z})_{\mathbb{L}^2} &= (\mathbf{f}, \mathbf{z})_{\mathbb{L}^2}, \text{ for all } \mathbf{z} \in V.
\end{align*}$$

Therefore, since $\text{div} \mathbf{y}_\gamma = 0$ and $\mathbf{z} \in V$, (4.16), (4.19) and (4.21) imply that

$$\begin{align*}
(\mu \Delta \mathbf{y}_\gamma + \nabla \mathbf{q}_\gamma + \mathbf{f}, \mathbf{z})_{H^{-1}(\Omega), H^1_0(\Omega)} &= 0,
\end{align*}$$

which, due to the de Rham’s Theorem (see [42, Rem. 1.9]), implies the existence of $p \in \mathbb{L}^2(\Omega)$ such that

$$\begin{align*}
-\mu \Delta \mathbf{y}_\gamma - \nabla \mathbf{q}_\gamma + \nabla p &= \mathbf{f} \text{ in } H^{-1}(\Omega).
\end{align*}$$

Moreover, [17, Th. 3.6, p. 34] implies the existence of a unique $p_1 \in L^2(\Omega) := \{ p \in \mathbb{L}^2(\Omega) : \int_{\Gamma_N} p(x) \, ds = 0 \}$ satisfying (4.22).

In addition, since $\mathbf{f} \in \mathbb{L}^2(\Omega)$, (4.22) yields that $-\mu \Delta \mathbf{y}_\gamma - \nabla \mathbf{q}_\gamma + \nabla p \in \mathbb{L}^2(\Omega)$. Since $\text{div} \mathbf{y}_\gamma = 0$, identity (4.18) then implies that

$$\begin{align*}
\mu \Delta \mathbf{y}_\gamma + \nabla \mathbf{q}_\gamma - \nabla p &= \text{Div}(2\mu \mathcal{E} \mathbf{y}_\gamma + \mathbf{q}_\gamma - p I) \in \mathbb{L}^2(\Omega),
\end{align*}$$

with $I$ denoting the $d \times d$ identity matrix. By multiplying (4.23) by $\mathbf{v} \in Y_{0,D}$ and integrating by parts (see [35, p. 11]), we get that

$$\begin{align*}
2\mu \int_{\Omega} \mathcal{E} \mathbf{y}_\gamma : \mathcal{E} \mathbf{v} \, dx + \int_{\Omega} \mathbf{q}_\gamma : \mathcal{E} \mathbf{v} \, dx - \int_{\Omega} p I : \mathcal{E} \mathbf{v} \, dx - \int_{\Omega} \langle \mathbf{f}, \mathbf{v} \rangle \, dx
- \int_{\Gamma_N} ((2\mu \mathcal{E} \mathbf{y}_\gamma + \mathbf{q}_\gamma - p I) \cdot \mathbf{n}, \mathbf{v}) \, ds &= 0, \text{ for all } \mathbf{v} \in Y_{0,D},
\end{align*}$$

which, since $(\mathbf{y}_\gamma, \mathbf{q}_\gamma)$ satisfy equation (4.6) and due to the fact that $p I : \mathcal{E}(\mathbf{v}) = p \text{div} \mathbf{v}$, yields that

$$\begin{align*}
(2\mu \mathcal{E} \mathbf{y}_\gamma + \mathbf{q}_\gamma - p I) \cdot \mathbf{n} = 0, \text{ in } \Gamma_N.
\end{align*}$$

Now, we can prove that $p$ is unique. Assume that there exists another $\tilde{p} \in \mathbb{L}^2(\Omega)$, which satisfies (4.22). Since $p_1 \in L^2(\Omega)$ uniquely satisfies (4.22), we can conclude that there exist $C_1, C_2 \in \mathbb{R}$ such that $p = p_1 + C_1$ and $\tilde{p} = p_1 + C_2$ (see [42] and [17, Rem. 3.5, p. 34]). Further, we have that

$$\begin{align*}
(2\mu \mathcal{E} \mathbf{y}_\gamma + \mathbf{q}_\gamma - p I) \cdot \mathbf{n} = 0 \text{ and } (2\mu \mathcal{E} \mathbf{y}_\gamma + \mathbf{q}_\gamma - \tilde{p} I) \cdot \mathbf{n} = 0, \text{ in } \Gamma_N.
\end{align*}$$
Consequently, we obtain that
\[(C_1 - C_2)\vec{n} = 0, \quad \text{on } \Gamma_N,\]
which, since \(\vec{n} \neq 0\), implies that \(C_1 = C_2\).

Let us now introduce the subspace \(X_{0,D} \subset H^1(\Omega)\) and the set \(X_D \subset H^1(\Omega)\) by
\[X_{0,D} := \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v} = 0 \text{ on } \Gamma_0 \cup \Gamma_D \},\]
\[X_D := \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v} = 0 \text{ on } \Gamma_0 \text{ and } \mathbf{v} = y_D \text{ on } \Gamma_D \}.

Thanks to Proposition 4.2, we obtain the following variational problem: find \((y_\gamma, q_\gamma, p) \in X_D \times K \times L^2(\Omega)\), such that
\[
\begin{cases}
 a(y_\gamma, \mathbf{v}) + (q_\gamma, \mathbf{v})_{L^2} - (p, \text{div} \mathbf{v})_2 = (f, \mathbf{v})_{L^2}, & \text{for all } \mathbf{v} \in X_{0,D} \\
 (r, \text{div} y_\gamma)_2 = 0, & \text{for all } r \in L^2(\Omega) \\
 q_\gamma(x) = \frac{\gamma}{\max(\gamma_1,\gamma_2)} y_\gamma(x), & \text{a.e. in } \Omega,
\end{cases}
\]
\((S_{\gamma,p})\)

Note that, if \((y_\gamma, q_\gamma, p) \in X_D \times K \times L^2(\Omega)\) solve problem \((S_{\gamma,p})\), \(y_\gamma \in Y_D\) and, since \(Y_0,0 \subset X_0,D\), \((y_\gamma, q_\gamma) \in Y_D \times K\) solve the optimality system \((S_\gamma)\). This fact, together with Proposition 4.2, implies that \((S_\gamma)\) and \((S_{\gamma,p})\) are equivalent.

5. DISCRETIZATION OF THE OPTIMALITY SYSTEMS

In this section, we utilize a finite element method in order to obtain a discretized version of the optimality system \((S_\gamma,p)\). Following the results in [35, p. 308], we first define a quadrangulation \(Q^h\) of \(\Omega\) made of closed squares. Then, from any square \(Q\) (macroelement) of \(Q^h\), we obtain four triangles by means of the two main diagonals of \(Q\). The obtained triangles define a triangulation \(T^h\) of \(\Omega\). The resulting elements are the so called (cross-grid \(P_1\)-Q0 elements, which correspond to taking piecewise linear velocities in each triangle and piecewise constant pressures in each square (see Figure 1). The associated spaces are stable (see [35, Sec. 9.3]) and lead to a direct relation between the discrete primal and dual variables. Therefore, we propose to take piecewise constant multipliers in each triangle, according to Figure 1.

5.1. Finite element approximation of system \((S_{\gamma,p})\).

In what follows, we focus on the case \(d = 2\) and identify the space \(L^{2\times 2}(\Omega)\) with the space \((L^2(\Omega))^4\). Indeed, let us define the operator \(\zeta : L^{2\times 2}(\Omega) \to (L^2(\Omega))^4\) by
\[
\zeta(\mathbf{q}) = (q_{11}, q_{12}, q_{21}, q_{22})^T, \quad \text{for } \mathbf{q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.
\]

Since the operator \(\zeta\) is clearly bijective, it is an isomorphism between \(L^{2\times 2}(\Omega)\) and \((L^2(\Omega))^4\). Let us recall that the space \((L^2(\Omega))^4\) is endowed with the usual \(L^2(\Omega)\)-scalar product, i.e.,
\[
(\mathbf{q}, \mathbf{p})_{(L^2)^4} := \int_\Omega (\mathbf{q}(x), \mathbf{p}(x)) \, dx.
\]

In such a case, we have that
\[
(\mathbf{q}, \mathbf{p})_{L^{2\times 2}} = \int_\Omega (\mathbf{p} : \mathbf{q}) \, dx = \int_\Omega (\zeta(\mathbf{p}), \zeta(\mathbf{p})) \, dx = (\zeta(\mathbf{q}), \zeta(\mathbf{p}))_{(L^2)^4}.
\]
Now, we construct an approximated version of system \((S_{\gamma,p})\) with (cross-grid \(P_1\))-\(Q_0\) elements. First, we define the finite dimensional Hilbert spaces \(V^h \subset H^1(\Omega), W^h \subset (L^2(\Omega))^4\) and \(M^h \subset L^2(\Omega)\) by

\[
V^h := (V_h \cap H^1(\Omega))^2, \quad W^h := \{v^h \in C(\Omega) : v^h|_T \in \Pi_1, \text{ for all } T \in T^h\},
\]

\[
W^h := (q^h_1, q^h_2, q^h_3, q^h_4) \in (L^2(\Omega))^4 : q^h_j|_T \in \Pi_0, \text{ for } j = 1, \ldots, 4 \text{ and } T \in T^h',
\]

\[
M^h := (h_0 \cap L^2(\Omega), \text{ where } h_0 := \{v^h \in C(\Omega) : v^h|_Q \in \Pi_0, \text{ for all } Q \in \mathcal{Q}_0^h\},
\]

where \(\Pi_k\) stands for the space of all polynomials defined on \(\mathbb{R}^N\) of degree less than or equal to \(k\). Here, \(Q^h\) and \(T^h\) are a regular quadrangulation and a regular triangulation of \(\Omega\), respectively (see [2, Sec. 4]). In order to simplify the analysis, we assume that \(\Omega\) has a polygonal boundary. Further, we state that \(\dim V^h = 2n, \dim W^h = 4m\) and \(\dim M^h = l\), with \(n, m, l \in \mathbb{N}\).

We also introduce the following subspaces of \(V^h\)

\[
X^h_{0,D} := \{v^h \in V^h : v^h = 0 \text{ on } \Gamma_0 \cup \Gamma_D\},
Y^h_{0,D} := \{v^h \in X^h_{0,D} : (r^h, \text{div} v^h)_2 = 0 \text{ for all } r^h \in M^h\},
\]

and the following convex sets of \(V^h\):

\[
X^h_D := \{v^h \in V^h : v^h = 0 \text{ on } \Gamma_0 \text{ and } v^h = y^h_D \text{ on } \Gamma_D\},
Y^h_D := \{v^h \in X^h_D : (r^h, \text{div} v^h)_2 = 0 \text{ for all } r^h \in M^h\},
\]

where \(y^h_D \in H^{1/2}_0(\Gamma_D)\) is an approximation of \(y_D \in H^{1/2}_0(\Gamma_D)\).

By using the last definitions, we can define a finite element discretization of \((S_{\gamma,p})\) by the following problem: find \(y^h \in X^h_{0,D}, q^h \in W^h\) and \(p^h \in M^h\), such that

\[
(5.1a) \quad a(y^h, v^h) + (q^h, \mathcal{E} v^h)_{(L^2)^4} - (p^h, \text{div} v^h)_2 = (f, v^h)_{L^2(\Omega)}, \quad \text{for } v^h \in X^h_{0,D},
\]

\[
(5.1b) \quad (r^h \text{div} y^h)_2 = 0, \quad \text{for } r^h \in M^h,
\]

\[
(5.1c) \quad \max(\bar{\gamma}, \gamma \|\mathcal{E} y^h(x)\|_{L^2})\|q^h(x)\|_{L^2} - \bar{\gamma} \gamma \mathcal{E} y^h(x) = 0, \text{ a.e. in } \Omega \text{ and } \gamma > 0.
\]

**Proposition 5.1.** The problem \((5.1)\) has a unique solution \((y^h, q^h, p^h) \in V^h \times W^h \times M^h\).

**Proof.** By using a similar argumentation as in Section 3, i.e., by using the Fenchel’s duality theory with the analogous discrete spaces and convex sets, it follows that there exists a unique pair \((y^h, q^h) \in Y^h_D \times W^h\) such that

\[
(5.2) \quad a(y^h, z^h) + (q^h, \mathcal{E} z^h)_{(L^2)^4} = (f, z^h)_{L^2(\Omega)}, \quad \text{for all } z^h \in Y^h_{0,D}
\]

\[
(5.3) \quad \max(\bar{\gamma}, \gamma \|\mathcal{E} y^h(x)\|_{L^2})\|q^h(x)\|_{L^2} - \bar{\gamma} \gamma \mathcal{E} y^h(x) = 0, \text{ a.e. in } \Omega \text{ and } \gamma > 0.
\]

Next, let \(A \in \mathcal{L}(V^h, (V^h)^*)\) be the linear operator associated to the bilinear form \(a\) by

\[
(\mathbf{Au}^h, v^h) = a(u^h, v^h), \quad \text{for all } u^h, v^h \in V^h
\]

(see [4, Th. 7.3.1]). Further, let us introduce the operator \(\hat{A} \in \mathcal{L}(X^h_{0,D}, W^h)\) by \(\hat{A} v^h := \mathcal{E} v^h\). Clearly, \(\hat{A}^* \in \mathcal{L}(W^h, (X^h_{0,D})^*)\). Here, we have identified \(W^h\) with its dual. Thus, from (5.2), we can conclude that

\[
\langle A y^h + \hat{A} q^h - f, z^h \rangle_{(X^h_{0,D})^*, X^h_{0,D}} = 0, \quad \text{for all } z^h \in Y^h_{0,D}.
\]

Therefore, since the (cross-grid \(P_1\)) - \(Q_0\) are stable (see [35, p. 308] and the references therein), they satisfy the LBB-condition. Thus, [17, Lem. 4.1, p. 40] implies the existence of a unique \(p^h \in M^h\), such that \((y^h, q^h, p^h)\) satisfy the equation (5.1a) and thus the entire system (5.1).

Next, by following the standard procedure, we can state that (5.1a) and (5.1b) are equivalent to the following system of equations

\[
(5.4a) \quad A^h y^h + Q^h q^h + B^h p^h = f^h,
\]

\[
(5.4b) \quad -(B^h)^\top y^h = 0.
\]

As always, the elements of the vectors \(\bar{y}^h \in \mathbb{R}^{2n}, \bar{q}^h \in \mathbb{R}^{4m}\) and \(\bar{p}^h \in \mathbb{R}^l\) are the coefficients in the representation in terms of bases of the finite element triplet \((y^h, q^h, p^h)\). \(A^h \in \mathbb{R}^{2n \times 2n}\) is the stiffness matrix, and the matrices \(Q^h \in \mathbb{R}^{2n \times 4m}\) and \(B^h \in \mathbb{R}^{2n \times l}\) are obtained in the usual way, from the bilinear forms \((\cdot, \cdot)_{(L^2)^4}\) and \(- \langle \cdot, \text{div} \cdot \rangle_2\), using the bases for \(V^h, W^h\) and \(M^h\).
respectively. The right hand side \( f^h \) is constructed by using the basis functions \( \varphi_j \in V^h, j = 1, \ldots, 2n \) (see [3, Sec. 6]). Hereafter, the matrix \( A^h \) is assumed to be symmetric and positive definite.

Next, we construct an equivalent formulation of (5.1c) in terms of the vectors \( \vec{y} \) and \( \vec{q} \) of coefficients of \( y^h \) and \( q^h \), respectively. First note that it is possible to state that \( \varphi_j := (\varphi_j e_1, \varphi_j e_2)^\top \), where \( \varphi_j, j = 1, \ldots, n \), are the scalar hat functions of the node \( j \) in the triangulation \( T^h \), and \( e_1, e_2 \) stand for the canonical vectors of \( \mathbb{R}^2 \). Thus, we propose a discrete version of \( E \) composed by the matrices \( \partial_{h1} := \left. \frac{\partial \varphi_j(x)}{\partial x_1} \right|_{T_k} \) and \( \partial_{h2} := \left. \frac{\partial \varphi_j(x)}{\partial x_2} \right|_{T_k} \), for \( j = 1, \ldots, n \) and \( k = 1, \ldots, m \), where \( \left. \frac{\partial \varphi_j(x)}{\partial x_1} \right|_{T_k} \) and \( \left. \frac{\partial \varphi_j(x)}{\partial x_2} \right|_{T_k} \) are the constant values of \( \left. \frac{\partial \varphi_j(x)}{\partial x_1} \right|_{T_k} \) and \( \left. \frac{\partial \varphi_j(x)}{\partial x_2} \right|_{T_k} \) in each triangle \( T_k \), respectively, i.e.,

\[
E^h = \frac{1}{2} \begin{bmatrix} 0_{m \times n} & \partial_{h1}^T & \partial_{h2}^T \\ \partial_{h1} & 0_{m \times n} & \partial_{h2} \\ \partial_{h2} & \partial_{h1} & 0_{m \times n} \end{bmatrix} \in \mathbb{R}^{4m \times 2n}.
\]

Clearly we have that \( E^h \vec{y} = \mathcal{E} y^h \). Further, in order to write \( \| \mathcal{E} y^h(x) \| \) in terms of \( \vec{y} \), we define the function \( N : \mathbb{R}^{4m} \rightarrow \mathbb{R}^{4m} \) by

\[
N(\vec{q}) = N(\vec{q})_{i+1} = \cdots = \left. N(\vec{q})_{i+4m} := \left[ q_i, q_{i+m}, \ldots, q_{i+4m} \right]^\top \right|_{\mathbb{R}^{4m}}
\]

for \( \vec{q} \in \mathbb{R}^{4m} \) and \( i = 1, \ldots, m \).

Finally, by using the last definitions, we can state that equation (5.1c) is equivalent to

\[
(5.5) \quad \max \left( g \vec{e}, \gamma N(E^h \vec{y}) \right) \ast \vec{q} - \gamma g \vec{e}^h \vec{y} = 0,
\]

and the discrete analogous of system \( (S_{\gamma,p}) \) is given by

\[
(S_{\gamma,p}) \begin{cases} \begin{aligned} A^h \vec{y} + Q^h \vec{q} + B^h \vec{g} - \vec{f} &= 0 \\ -(B^h)^\top \vec{y} &= 0 \\ \max \left( g \vec{e}, \gamma N(E^h \vec{y}) \right) \ast \vec{q} &= \gamma g \vec{e}^h \vec{y}, \end{aligned} \end{cases}
\]

where \( \vec{e} \in \mathbb{R}^{4m} \) denotes the vector of all ones.

**Remark 5.1.** Let \( T^h \in \mathbb{R}^{m \times m} \) be a diagonal matrix with the values of the areas of each triangle \( T_k \in T^h \), \( k = 1, \ldots, m \), in its diagonal. Thus, note that we can always rewrite matrix \( Q^h \) as

\[
Q^h = \left( Q^h_1, Q^h_2, Q^h_3, Q^h_4 \right),
\]

where \( Q^h_\ell \in \mathbb{R}^{2n \times m} \), for \( \ell = 1, \ldots, 4 \). Moreover, we have that

\[
Q^h_\ell = \left( (Q^h_\ell)^1, (Q^h_\ell)^2 \right), \quad \text{where} \quad (Q^h_\ell)^1 \in \mathbb{R}^{n \times m} \text{ and } (Q^h_\ell)^2 \in \mathbb{R}^{n \times m}.
\]

Next, let \( (Q^h_\ell)^1_j \), \( j = 1, \ldots, n \), denote each row of the matrix \( (Q^h_\ell)^1 \) and let \( (Q^h_\ell)^2_j \), \( j = 1, \ldots, n \), denote each row of the matrix \( (Q^h_\ell)^2 \). Further, let \( (Q^h_k)^1 \in \mathbb{R}^{m \times 1} \) and \( (Q^h_k)^2 \in \mathbb{R}^{m \times 1} \) represent each column of the submatrices in the matrix \( E^h \). Due to the selection of the finite element spaces \( V^h \) and \( W^h \), we have the following representation for matrices \( (Q^h_\ell)^1 \) and \( (Q^h_\ell)^2 \).

- If \( \ell = 1 \), \( (Q^h_1)^1_k = \left[ (\partial_{h1}^k)^T h^T \right] \), for \( k = 1, \ldots, m \), and \( (Q^h_1)^2 \) is \( 0_{1 \times m}. \)
- If \( \ell = 2 \), \( (Q^h_2)^1_k = \left[ (\partial_{h2}^k)^T h^T \right] \) and \( (Q^h_2)^1_k = \left[ (\partial_{h1}^k)^T h^T \right] \), for \( k = 1, \ldots, m. \)
- If \( \ell = 3 \), \( (Q^h_3)^1_k = \left[ (\partial_{h3}^k)^T h^T \right] \) and \( (Q^h_3)^1_k = \left[ (\partial_{h2}^k)^T h^T \right] \), for \( k = 1, \ldots, m. \)
- If \( \ell = 4 \), \( (Q^h_4)^1_k = 0_{1 \times m} \) and \( (Q^h_4)^1_k = \left[ (\partial_{h3}^k)^T h^T \right] \), for \( k = 1, \ldots, m. \)

Clearly, if we choose a uniform triangulation, where all the triangles are equal, we have that

\[
Q^h = |T| (E^h)^T,
\]

where \( |T| \) represents the constant area of any triangle in \( T^h \).
In this section we propose a semismooth Newton method for the solution of the approximated system \((\mathcal{S}^h_{\gamma,p})\). The algorithm proposed here is based on the Newton type algorithm developed in [26]. Hereafter, for a vector \(\vec{v} = (v_1, \ldots, v_n)^T\), we denote by \(D(\vec{v})\) the \(n \times n\)-diagonal matrix, \(n \in \mathbb{N}\), whose diagonal entries are given by \(v_i\).

6.1. **Approximated semismooth Newton step.** Following [26], we construct a semismooth Newton algorithm for the solution of system \((\mathcal{S}^h_{\gamma,p})\). Similarly as in [9], we look for a decoupled system of linear equations to be solved in each Newton iteration, whose solutions constitute descent directions for the objective functional in \((\mathcal{P}_\gamma)\). The idea is to penalize the equation (5.4b) in order to relax the incompressibility constraint, by taking the following constraint instead

\[
(B^h)^T \tilde{y} + \varsigma \vec{p} = 0,
\]

with \(\varsigma > 0\) sufficiently small. This kind of penalization is motivated by stabilization procedures developed for the finite element approximation of Stokes and Navier-Stokes equations (see [5, 23]). In our case, this procedure leads to a decoupled system of linear equations in each Newton iteration (see equations (6.4)-(6.6) below).

Next, we construct an algorithm to efficiently solve the system \((\mathcal{S}^h_{\gamma,p})\) considering (5.4b') instead of (5.4b). This system can be written as the following operator equation:

\[
F(\tilde{y}, \tilde{q}, \tilde{p}) := \begin{bmatrix} A^h_{\beta} \tilde{y} + Q^h \tilde{q} + B^h \tilde{p} - \vec{f} \\ -(B^h)^T \tilde{y} + \varsigma \tilde{p} \\ D(m^k_h) \tilde{q} - \gamma \tilde{g} \tilde{c}^h \tilde{y} \end{bmatrix} = 0,
\]

where \(m^k_h := \max(\tilde{g}c(R\tilde{y}_k)) \in \mathbb{R}^{4m}\). It is well known that the max-operator and the norm function \(N\) involved in (6.1) are semismooth. This is also true for the composition of semismooth functions that arises in (6.1) (see, for instance, [24, 26, 40]). Consequently, we are able to calculate the Newton step to (6.1), at the approximations \((\tilde{y}, \tilde{q}, \tilde{p})\), by:

\[
\begin{bmatrix} A^h_{\beta} \tilde{y} + Q^h \tilde{q} + B^h \tilde{p} - \vec{f} \\ -(B^h)^T \tilde{y} + \varsigma \tilde{p} \\ D(m^k_h) \tilde{q} - \gamma \tilde{g} \tilde{c}^h \tilde{y} \end{bmatrix} = \begin{bmatrix} \delta y \\ \delta p \\ \delta q \end{bmatrix}
\]

\[
= -A^h_{\beta} \tilde{y}_k - Q^h \tilde{q}_k - B^h \tilde{p}_k + \vec{f}
\]

\[
= \begin{bmatrix} D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \end{bmatrix}^{-1} \begin{bmatrix} D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \end{bmatrix} \begin{bmatrix} \delta y \\ \delta p \\ \delta q \end{bmatrix}.
\]

6.2. where \(\chi_A = D(t^k_h) \in \mathbb{R}^{4m \times 4m}\) with

\[
(t^k_h)_i := \begin{cases} 1 & \text{if } N(\tilde{c}^h \tilde{y}_k)_i \geq \frac{\tilde{g}}{\gamma} \\ 0 & \text{else} \end{cases}
\]

\(N^h_{\tilde{c}}\) denotes the Jacobian of \(N\), and is given by

\[
N^h_{\tilde{c}}(q) = (D(\tilde{c}(q)))^{-1} \begin{bmatrix} D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \\ D(q_1) & D(q_2) & D(q_3) & D(q_4) \end{bmatrix},
\]

for \(q = (q_1, q_2, q_3, q_4)^T \in \mathbb{R}^{4m}\), with \(q_1, q_2, q_3, q_4 \in \mathbb{R}^m\).

Now, we analyze system (6.2). First, note that, since \(D(m^k_h)\) is invertible and \(\varsigma \neq 0\), we obtain that

\[
\delta q := -\tilde{q}_k + (D(m^k_h))^{-1} (\gamma \tilde{g} \tilde{c}^h \tilde{y}_k + C^h_{\beta} \tilde{c}^h \delta y),
\]

\[
\delta p := \frac{1}{\varsigma} ((B^h)^T \tilde{y}_k + (B^h)^T \delta y) - \tilde{p}_k,
\]

where the matrix \(C^h_{\beta}\) is given by

\[
C^h_{\beta} := \gamma I_{4m} - \gamma \chi_A \delta_{k+1} D(q_k) N^h_{\tilde{c}}(\tilde{c}^h \tilde{y}_k).
\]
Thus, we only have the following remaining equation for $\delta y$

$$
(6.6) \quad \left[ \Xi_{\gamma,k} + \frac{1}{\varsigma} B^h(B^h)^\top \right] \delta y = \eta_{\gamma,k} - \frac{1}{\varsigma} B^h(B^h)^\top \bar{y}_k
$$

where the matrix $\Xi_{\gamma,k}$ and the vector $\eta_{\gamma,k}$ are defined by

$$
\Xi_{\gamma,k} := A^h_i + Q^h(D(m_i^h))^{-1} C_k^h \mathcal{E}^h
$$

$$
\eta_{\gamma,k} := -A^h_i \bar{y}_k + \bar{f} - \bar{g} \gamma Q^h(D(m_i^h))^{-1} \mathcal{E}^h \bar{y}_k
$$

Next, we analyze the properties of matrix $\Xi_{\gamma,k}$. Clearly, if we guarantee the positive definiteness of this matrix, we also guarantee the positive definiteness of the whole system matrix $[\Xi_{\gamma,k} + \frac{1}{\varsigma} B^h(B^h)^\top]$, and we can assure that in each generalized Newton step a descent direction is calculated. Thus, in the following analysis we look for conditions which guarantee positive definiteness of $\Xi_{\gamma,k}$, for all $\gamma$ and $k$.

We start by studying properties of $C_k^h$. Following the ideas of [26], we can reorder the indices in such a way that the matrix $D(q_k)\mathcal{N}_k^h(\mathcal{E}^h y_k^h)$ becomes a block-diagonal matrix, where every $4 \times 4$ diagonal block has the form

$$
(w_k^h)_{ij} := \begin{cases} 
q_k, & \text{if } (q_k)_{ij} = 1 \\
0, & \text{otherwise}
\end{cases}
$$

for $i = 1, \ldots, n$. Due to this reordering, $C_k^h$ is transformed into a block-diagonal matrix with the following diagonal blocks

$$
(6.7) \quad (c_k^h)_{ij} := \bar{g} I_4 - (t_k^h)_{ij} (w_k^h)_{ij},
$$

where $(t_k^h)_{ij} \in \{0, 1\}$ are given by (6.3) and $I_4$ stands for the $4 \times 4$ identity matrix. Since $(c_k^h)_{ij}$ is obviously positive definite for inactive indices i.e., for all $i$ with $(t_k^h)_{ij} = 0$, we look for conditions which implies that $(c_k^h)_{ij}$ is positive definite for indices in which $(t_k^h)_{ij} = 1$. Now, since positive definiteness of the symmetric part of matrix $(c_k^h)_{ij}$ implies positive definiteness of the matrix itself (see [30, Rem. 1]), we analyze the following matrix

$$
(c_k^h)_{ij} := \frac{(c_k^h)_{ij} + (c_k^h)_{ji}}{2} = \begin{bmatrix}
\frac{a_{k,i}}{2} & \frac{b_{k,i} + c_{k,i}}{2} & \frac{b_{k,i} + m_{k,i}}{2} & \frac{d_{k,i} + r_{k,i}}{2} \\
\frac{b_{k,i} + r_{k,i}}{2} & \frac{f_{k,i}}{2} & \frac{h_{k,i} + n_{k,i}}{2} & \frac{l_{k,i} + s_{k,i}}{2} \\
\frac{b_{k,i} + m_{k,i}}{2} & \frac{h_{k,i} + n_{k,i}}{2} & \frac{p_{k,i} + s_{k,i}}{2} & \frac{u_{k,i}}{2} \\
\frac{d_{k,i} + r_{k,i}}{2} & \frac{l_{k,i} + s_{k,i}}{2} & \frac{p_{k,i} + s_{k,i}}{2} & \frac{u_{k,i}}{2}
\end{bmatrix}
$$
Moreover, from \([32, p. 2]\), we conclude that \(B\) is symmetric and assumed to be positive definite, and \(\Xi\) is also symmetric and positive definite, since the condition \(\lambda_{\min}(A_{\mu}^h) \geq 0\) holds for all \(k \in \mathbb{N}\) and \(A_{\mu}\) is symmetric and assumed to be positive definite, and \(B^h(B^h)^\top\) is also symmetric and positive semidefinite. Moreover, from [32, p. 2], we conclude that

\[
\lambda_{\min} \left( \Xi_{\gamma,k} + \frac{1}{\zeta} B^h(B^h)^\top \right) \geq \lambda_{\min}(\Xi_{\gamma,k}) \geq \lambda_{\min}(A_{\mu}^h) > 0.
\]

Finally, from [32, p. 3], we obtain

\[
\left\| \left( \Xi_{\gamma,k} + \frac{1}{\zeta} B^h(B^h)^\top \right)^{-1} \right\|_{sp} = \frac{1}{\lambda_{\min}(\Xi_{\gamma,k})} \leq \frac{1}{\lambda_{\min}(A_{\mu}^h)},
\]

where \(\| \cdot \|_{sp}\) states for the spectral norm of matrices.

**6.2. A modified semismooth Newton algorithm.** If we assume that the condition \(N(\bar{q}_k) \leq \bar{g}\) holds for all \(i = 0, \ldots, m\) and \(k \in \mathbb{N}\), Proposition 6.1 and Corollary 6.2 imply the existence, for all \(k \in \mathbb{N}\), of a unique solution for (6.6). Moreover, we have that this solution is a descent direction for the objective functional in (P_{\gamma}). However, the condition \(N(\bar{q}_k) \leq \bar{g}\) is unlikely to hold for all \(i = 1, \ldots, m\) and \(k \in \mathbb{N}\). To solve this problem, following [26], we modify the term involving \(D(\bar{q}_k)N_{\gamma}^h(\bar{B}^h\bar{y}_k)\) for indices \(i\) in which the condition \(N(\bar{q}_k)_i \leq \bar{g}\) is not fulfilled.

The idea is to replace \(((q_{k+1}^h)_i, (q_{k+1}^h)_{i+2m}, (q_{k+1}^h)_{i+3m})\) by

\[
g \max \left( g, N(\bar{q}_k)_i \right)^{-1} \left( (q_{k}^h)_i, (q_{k}^h)_{i+2m}, (q_{k}^h)_{i+3m} \right),
\]

when assembling the system matrix \(\Xi_{\gamma,k}\). Note that, due to this projection step, the resulting matrix remains positive definite, since the condition \(N(\bar{q}_k)_i \leq \bar{g}\) holds for all \(i = 1, \ldots, m\) and all
for $k \in \mathbb{N}$. Thus, revoking the reordering of the indices, we obtain a modified matrix, denoted by $\hat{\Xi}_{\gamma,k}$, which will be used, instead of $\Xi_{\gamma,k}$, in (6.6).

**Lemma 6.3.** The matrix $\left[ \hat{\Xi}_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top} \right]$ is uniformly positive definite, and $\lambda_{\min}(\hat{\Xi}_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top}) \geq \lambda_{\min}(A_{0}^{h}) > 0$. Moreover, the sequence

$$\left\{ \left( \hat{\Xi}_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top} \right)^{-1} \right\}_{k \in \mathbb{N}}$$

is uniformly bounded.

**Proof.** The proof follows in the same way as in the Corollary 6.2. \qed

Next, we present a modified semismooth Newton algorithm to solve (6.1). This algorithm works with $\hat{\Xi}_{\gamma,k}$ whenever the condition $N(\tilde{q}_{k})_{i} \leq \tilde{g}$ fails to be fulfilled, and with the actual system matrix $\Xi_{\gamma,k}$, when this condition is already satisfied.

**Algorithm (GSSN)**

1. Initialize $(\hat{y}_{0}, q_{0}, \tilde{p}_{0}) \in \mathbb{R}^{2n} \times \mathbb{R}^{4m} \times \mathbb{R}^{l}$ and set $k = 0$.
2. Estimate the active sets, i.e., determine $\chi_{4k+1} \in \mathbb{R}^{4m \times 4m}$.
3. Compute $\hat{\Xi}_{\gamma,k}$ if the dual variable is not feasible for all $i = 1, \ldots, l$; otherwise set $\hat{\Xi}_{\gamma,k} = \Xi_{\gamma,k}$.

Solve

$$\left[ \hat{\Xi}_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top} \right] \delta y = \eta_{\gamma,k} - \frac{1}{\varsigma} B^{h}(B^{h})^{\top} \tilde{y}_{k}.$$  

4. Compute $\delta q$ and $\delta p$ according to (6.4) and (6.5) respectively.
5. Update $\tilde{y}_{k+1} := \hat{y}_{k} + \delta y$, $\tilde{q}_{k+1} := \tilde{q}_{k} + \delta q$ and $\tilde{p}_{k+1} := \tilde{p}_{k} + \delta p$.
6. Stop, or set $k := k + 1$ and go to step 2.

**Lemma 6.4.** Let $(\hat{y}_{\gamma}, \hat{q}_{\gamma})$ be the solutions to (6.11), and assume that $\hat{y}_{k} \to \hat{y}_{\gamma}$ and $\hat{q}_{k} \to \hat{q}_{\gamma}$. Then, the modified matrices $\hat{\Xi}_{\gamma,k}$ converge to $\Xi_{\gamma,k}$ as $k \to \infty$.

**Proof.** This proof again uses the reordering of the indices used before. As above, we study every (4×4)-block matrices separately. Note that for inactive indices, i.e., for $i$ such that $(t_{i}^{h})_{i} = 0$, the original and the modified (4×4)-blocks coincide, and the results follows immediately.

Next, we turn to the active indices, i.e., to indices such that $(t_{i}^{h})_{i} = 1$. Due to the assumption $\tilde{q}_{k} \to \hat{q}_{\gamma}$, and to the continuity of the function $N$, we have that $(N(\tilde{q}_{k}))_{i} \to (N(\hat{q}_{\gamma}))_{i} \leq g$. This fact implies that

$$g \max \left( g, N(\tilde{q}_{k})_{i} \right)^{-1} ((\tilde{q}_{k})_{i}, (\tilde{q}_{k})_{i+1+m}, (\tilde{q}_{k})_{i+2+m}, (\tilde{q}_{k})_{i+3+m}) \to ((\tilde{q}_{\gamma})_{i}, ((\tilde{q}_{\gamma})_{i+1+m}, (\tilde{q}_{\gamma})_{i+2+m}, (\tilde{q}_{\gamma})_{i+3+m}, as k \to \infty.$$  

Moreover, there exists $k_{0} \in \mathbb{N}$ such that $(N(\tilde{q}_{k}))_{i} \leq \tilde{g}$ for all $k \geq k_{0}$. Hence, all modified (4×4)-diagonal blocks converge to the original blocks as $k \to \infty$. Revoking the reordering of the indices finishes the proof. \qed

Now we analyze the local convergence of the GSSN Algorithm. Let us recall that the equations involved in function (6.1) are semismooth. However, since we probably modify our system matrix, we obtain a semismooth quasi-Newton method (see [34, 41]). Therefore, fast local convergence follows from different arguments than the standard ones (see [26]).

**Theorem 6.5.** The iterates $(\hat{y}_{k}, \hat{q}_{k}, \tilde{p}_{k})$ of the algorithm converge superlinearly to $(\hat{y}_{\gamma}, \hat{q}_{\gamma}, \tilde{p})$ provided that $(\hat{y}_{0}, \hat{q}_{0}, \tilde{p}_{0})$ are sufficiently close to $(\hat{y}_{\gamma}, \hat{q}_{\gamma}, \tilde{p})$.

**Proof.** First, from Lemma (6.4), we have that $\hat{y}_{k} \to \hat{y}_{\gamma}$ and $\hat{q}_{k} \to \hat{q}_{\gamma}$ imply that

$$((\hat{\Xi}_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top}) \to ((\Xi_{\gamma,k} + \frac{1}{\varsigma} B^{h}(B^{h})^{\top}), as k \to \infty.$$  

Thus, Lemma 6.3 and (6.8) allow us to conclude that there exist constants $A_{1} > 0$, $A_{2} > 0$ and $\rho > 0$ such that

$$\|\hat{y}_{k}, q_{k}, \tilde{p}_{k} - (\hat{y}_{\gamma}, \hat{q}_{\gamma}, \tilde{p})\| \leq \rho$$  

for $k \geq k_{0}$.
implies that
\[ \| \Xi_{\gamma,k} - \hat{\Xi}_{\gamma,k} \| \leq A_1 \quad \text{and} \quad \| \hat{\Xi}_k \| \leq A_2. \]
Therefore, the assumptions of [41, Th. 4.1] are satisfied. This Theorem implies that the iterates \((\vec{y}_k, \vec{q}_k, \vec{p}_k)\) converge to the solution \((\vec{y}_\gamma, \vec{q}_\gamma, \vec{p})\) at a linear rate, provided that \((\vec{y}_0, \vec{q}_0, \vec{p}_0)\) is sufficiently close to \((\vec{y}_\gamma, \vec{q}_\gamma, \vec{p})\). Furthermore, (6.8) implies that the assumptions of [41, Th. 4.2] are also fulfilled. Theorem [41, Th. 4.2] guarantees that the convergence is at a superlinear rate, if \((\vec{y}_0, \vec{q}_0, \vec{p}_0)\) lies in a neighborhood of the solution \((\vec{y}_\gamma, \vec{q}_\gamma, \vec{p})\). Consequently, the algorithm GSSN is locally superlinearly convergent.

**Remark 6.1.** Steps 3 and 4 of the algorithm GSSN constitutes a decoupled system of equations for \(\delta_y\), \(\delta_q\) and \(\delta_p\), which is obtained, directly, due to the Tikhonov regularization and due to the penalization procedure (5.4b’). The algorithm only needs to solve a \(2n \times 2n\)-system of linear equations to calculate \(\delta_y\) (see (6.6)), the computation of \(\delta_q\) needs only the inverse of a diagonal matrix (see (6.4)), and the computation of \(\delta_p\) reduces to an assignment (see (6.5)).

**Remark 6.2.** The projection procedure, which let us construct the matrix \(\hat{\Xi}_{\gamma,k}\), guarantees that in each iteration of algorithm GSSN,
\[ \delta_y = \left( \frac{1}{\zeta} A_h^{\top} (B^h)^\top \right)^{-1} \left( \eta_{\gamma,k} - \frac{1}{\zeta} B^h (B^h)^\top \vec{y}_k \right) \]
is a descent direction for the objective functional in \((P_{\gamma})\). In the case that the penalization procedure (5.4b’) is not used, it is also possible to decouple system (6.2) into a Stokes type system for \(\delta_y\) and \(\delta_p\) and a similar expression as (6.4) for \(\delta_q\). This issue will be considered in future research.

7. Numerical Results

In this section, we present numerical experiments which illustrate the main properties of the GSSN algorithm applied to the numerical solution of the two proposed two-dimensional stationary Bingham fluid flow problems.

In the examples below, the algorithm is initialized with the solution of the following discrete Stokes problem
\begin{equation}
\begin{aligned}
A^h_{\gamma} \vec{y}_0 + B^h \vec{p}_0 &= f^h \\
-B^h \vec{y}_0 + \varsigma \vec{p}_0 &= 0
\end{aligned}
\end{equation}
(together with \(\vec{q}_0 = 0\)). Let \(\delta^h := ||\delta_y||_{H^{1,h}} + ||\delta_q||_{L^2,h} + ||\delta_p||_{L^2,h}\), where \(\| \cdot \|_{H^{1,h}}\), \(\| \cdot \|_{L^2,h}\) stand for the discrete versions of \(\| \cdot \|_{H^1}\), \(\| \cdot \|_{L^2}\) and \(\| \cdot \|_{L^2}\), respectively. We stop the algorithm GSSN as soon as \(\delta^h\) is lower than \(\sqrt{\epsilon}\), where \(\epsilon\) denotes the machine accuracy \((\approx 2.22046 \times 10^{-16})\). Additionally, we choose \(\epsilon := \sqrt{\epsilon}\). Further, we compute the experimental convergence rate
\[ \nu_k^h = \frac{||\vec{y}_{k+1} - \vec{y}_k||_{H^{1,h}}}{||\vec{y}_k - \vec{y}_{k-1}||_{H^{1,h}}} \]
in order to numerically verify the superlinear convergence of the method. We consider uniform meshes whose components have all the same area and measure the size of these meshes by the constant radius of the inscribed circumferences of the triangles in the mesh, represented by \(h\).

7.1. Flow in a driven cavity. Here, we compute the flow of a Bingham fluid in a wall-driven cavity in the unit square \(\Omega_b := (0,1) \times (0,1)\). We assume that \(\Gamma_D = (x_1, 1)\), with \(x_1 \in (0,1)\), \(\Gamma_0 = \Gamma \setminus \Gamma_D\) and \(\Gamma_N = 0\). We take \(h = 0.0021\) \((\approx 1/476)\) and analyze the flow of the Bingham fluid considering \(f = 0\) and the following Dirichlet boundary condition
\[ \vec{y}^h_D (x) = \begin{cases} 0 & \text{if } x \in \Gamma_0 \\ (1,0) & \text{if } x \in \Gamma_D. \end{cases} \]

Let us fix \(\gamma = 10^3\) and \(g = 1\). In Table 1, we show the numerical behavior of the algorithm. We verify the convergence of \(\vec{y}_k^h\) towards \(\vec{y}^h\) as \(k \to \infty\), since \(\delta^h\) converges to 0. Additionally, since \(\nu_k^h\) decreases in the last iterations the local superlinear convergence rate is also verified.
In Figure 2, we show the velocity vector field, the flow streamlines and the computed active and inactive sets. Here it is possible to observe the expected stagnation zones in the bottom of the cavity and the rigid zone in the upper part of the square due to the action of $y^h_D$.

In Table 2, we show the behavior of the algorithm $GSSN$ for different values of $g$. Particularly, we show the numerical approximation of the final active set $|A_\gamma|$, the final residual $\delta^h$ and the rate $\nu^h_k$. Note that this Table let us conclude that the behavior of the algorithm looks to be independent of the value of $g$.

Further, in Table 3, we compare the behavior of the algorithm for different and increasing values of $\gamma$. As expected, the number of iterations increases, and the system matrix tends to be ill conditioned for high values of this regularization parameter.

| # it. | $\delta^h$ | $\nu^h_k$ | $|A_{k+1}|$ |
|-------|-----------|-----------|-----------|
| 1     | 421.6459  | 421.6459  | 41504     |
| 2     | 89.3617   | 0.2119    | 41120     |
| 3     | 41.9714   | 0.4697    | 39880     |
| 4     | 30.6921   | 0.7313    | 37480     |
| 5     | 31.0374   | 1.0113    | 33568     |
| 6     | 45.8865   | 1.4784    | 31296     |
| 7     | 23.4515   | 0.5111    | 31004     |
| 8     | 1.7062    | 0.0728    | 31056     |
| 9     | 0.0184    | 0.0108    | 31056     |
| 10    | 1.4401e-5 | 7.8217e-4 | 31056     |
| 11    | 6.9504e-11| 4.8264e-6 | 31056     |

**Table 1.** Driven cavity: residual $\delta^h$, convergence rate $\nu^h_k$ and size of active set $|A_{k+1}|$.

| $g$ | # it. | $|A_\gamma|$ | $\delta^h$  | $\nu^h_k$ |
|-----|-------|--------------|-------------|-----------|
| 1   | 11    | 38184        | 4.854e-11  | 1.652e-5  |
| 5   | 11    | 26024        | 3.173e-10  | 1.012e-6  |
| 10  | 11    | 21208        | 7.243e-10  | 3.322e-6  |
| 15  | 12    | 18680        | 1.617e-10  | 1.377e-4  |
| 20  | 12    | 16992        | 8.823e-10  | 4.673e-4  |

**Table 2.** Driven cavity: $h = 0.0021$, $\mu = 1$, $\gamma = 10^3$. 

**Figure 2.** Driven cavity. Left: velocity vector field. Right: streamlines, active ($A_\gamma$, black) and inactive ($I_\gamma$, gray) sets. Parameters: $\mu = 1$, $g = 2.5$. 

In Figure 2, we show the velocity vector field, the flow streamlines and the computed active and inactive sets. Here it is possible to observe the expected stagnation zones in the bottom of the cavity and the rigid zone in the upper part of the square due to the action of $y^h_D$. 

In Table 2, we show the behavior of the algorithm $GSSN$ for different values of $g$. Particularly, we show the numerical approximation of the final active set $|A_\gamma|$, the final residual $\delta^h$ and the rate $\nu^h_k$. Note that this Table let us conclude that the behavior of the algorithm looks to be independent of the value of $g$.

Further, in Table 3, we compare the behavior of the algorithm for different and increasing values of $\gamma$. As expected, the number of iterations increases, and the system matrix tends to be ill conditioned for high values of this regularization parameter.
Table 3. Driven cavity. For each value $\gamma$: number of iterations, final residual $\delta^h$ and final convergence rate $\nu^h_k$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th># it.</th>
<th>$\delta^h$</th>
<th>$\nu^h_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3.9195e-11</td>
<td>7.8631e-7</td>
</tr>
<tr>
<td>$10^2$</td>
<td>10</td>
<td>7.3054e-11</td>
<td>2.0609e-6</td>
</tr>
<tr>
<td>$10^3$</td>
<td>11</td>
<td>6.9504e-11</td>
<td>4.8264e-6</td>
</tr>
<tr>
<td>$10^6$</td>
<td>42</td>
<td>4.8379e-10</td>
<td>9.2603e-4</td>
</tr>
</tbody>
</table>

Figure 3. Bounded channel: computational domain

7.2. Flow in a bounded channel. In this example we compute the flow of a Bingham fluid, defined by $\mu = 1$ and $g = 15$, in the geometry specified in Figure 3. We assume that $\Gamma_0 = \Gamma \setminus (\Gamma_D \cup \Gamma_N)$ and that $\text{meas}(\Gamma_N) \neq 0$. In $\Gamma_D$, we take

\begin{align*}
    y_D(x) := \begin{pmatrix} y_{D,1} \\ y_{D,2} \end{pmatrix} = \begin{pmatrix} -(4/9)x_2^2 + (4/3)x_2 \\ 0 \end{pmatrix},
\end{align*}

and on $\Gamma_N$ we impose a stress-free or “do nothing” boundary condition of the type $\sigma_{Tot} \cdot \vec{n} = 0$ in $\Gamma_N$, where $\sigma_{Tot}$ is the total Cauchy stress tensor and $\vec{n}$ is the outward normal vector (see [43]). We also assume that there is not a forcing term, i.e., $f = (0, 0)^\top$, and the material is expected to flow just under the effect of the parabolic inflow $y_D$. In Table 4, we show the convergence behavior of the algorithm with $h = 0.0061$, (≈ 1/164), and $\gamma = 10^3$. As in the previous examples, we document the fast local convergence by showing the values of the norm of the residual $\delta^h$ and the rate $\nu^h_k$. We observe a fast decrease of the two rates at the end of the iterations, which confirms the superlinear local convergence of the algorithm. Further, we depict the number of triangles where the dual iterates $\vec{q}_k$ are infeasible (\# $|\vec{q}_k| > \tilde{g}$), i.e., triangle in which $|\vec{q}_k| > (1 + \theta)\tilde{g}$, where $\theta$ is a correction factor introduced to avoid roundoff errors ($\theta \approx 10^{-7}$). The algorithm GSSN allows the dual variable to be infeasible in each iteration. Note that the algorithm uses iterates that violate the feasibility condition imposed in the dual variable, $|\vec{q}_k| \leq \tilde{g}$, but at the end the number of components of the dual variable which violate the dual constraint tends to zero as the algorithm approaches the solution (see [26]).

Let us now turn to the analysis of the mechanical properties of the flow. In Figure 4, the velocity vector field of the flow is depicted, while in Figure 5 the streamlines and the final active and inactive sets are shown. The inactive set corresponds to the triangles in which $\gamma(N(\mathcal{E}^h\mathbf{y}^h_i))_i \leq \tilde{g}$. The computed solution exhibits the characteristic properties of Bingham materials, e.g., it is possible to observe stagnant zones in the corners of the channel. In the final part of the geometry the fluid describes a laminar flow and the stress transmitted by the layers in this region decreases toward the channel center. Thus, the fluid behaves like a solid in this region. This behavior can be observed in Figure 5. In Table 5, we show the behavior of the algorithm for different and decreasing sizes of the mesh. The number of iterations for different mesh size steps is given in the second column.

Finally, In Table 6 we show the number of iterations compared with different initializations for the algorithm GSSN. Note that the algorithm requires only one additional iteration compared to the initialization given by the Stokes problem in (7.1).
$\#$ | $\delta^h$ | $\nu^h_k$ | $\#|q_k| > g$
---|---|---|---
1 | 4.9625e+3 | 4.9625e+3 | 0
2 | 3.1985e+3 | 0.6445 | 2620
3 | 479.3978 | 0.1499 | 15960
4 | 288.2611 | 0.6013 | 22136
5 | 273.0983 | 0.9474 | 22748
6 | 91.4509 | 0.4471 | 2688
7 | 40.8915 | 0.4471 | 2688
8 | 4.1830 | 0.1023 | 652
9 | 0.3599 | 0.0860 | 160
10 | 4.3779e-4 | 0.0012 | 0
11 | 3.7026e-9 | 8.4574e-6 | 0

Table 4. Bounded channel: norm of residual $\delta^h$, convergence rate $\nu^h_k$ and number of components of the dual variable which violate the dual constraint $\#|q_k|$

Figure 4. Bounded channel: Magnitude of the velocity of the flow. $\mu = 1$ and $g = 15$.

Figure 5. Bounded channel: streamlines and inactive $I_\gamma$ set (white). $\mu = 1$ and $g = 15$. 
The proposed semismooth Newton algorithm converges locally with a superlinear convergence rate. Our approach also guarantees that a descent direction for the objective functional in \((\mathcal{P}_*)\) is obtained in each iteration, leading to a global convergent behavior of algorithm \textbf{GSSN}. The theoretical results are computationally verified by means of two detailed numerical experiments.

In addition, thanks to the proposed methodology we obtain a decoupled system of equations (6.2) to be solved in each iteration of the \textbf{GSSN} algorithm. Therefore, the computation of the residuals \(\delta_q, \delta_i\) and \(\delta_p\) is carried out at low computational cost (see Remark 6.1). For the numerical experiments under investigation the algorithm needs, in average, 11 iterations to converge, which implies that, in average, only 11 \(2n \times 2n\)-systems of linear equations need to be solved to achieve convergence (see Tables 1, 2, 4 and 5).

Finally, let us mention that the analytical results obtained in this paper are also valid for the three-dimensional case and an extension of the proposed algorithm to deal with the 3D case will be considered in future research.

### APPENDIX

**Proof of Proposition 6.1.** Let us start by recalling the Sylvester criterion, which states that \((\mathcal{E}_k^b)_{ij}\) is positive definite if and only if all of the leading principal minors are positive, i.e., if and only if \(\det(\mathcal{E}_k^b)_{ij} \geq 0\), for \(j = 1, \ldots, 4\), where \((\mathcal{E}_k^b)_{ij}\) stands for the upper left \(j \times j\) corner of \((\mathcal{E}_k^b)_{ij}\).

Note that \(a_{i,k} \geq 0\) directly follows from the assumption \(N(\bar{q}_k)_i \leq \bar{g}\) and from \(\frac{\langle \bar{q}_k^b, \bar{w} \rangle}{\gamma} \leq 1\), and so we obtain that \(\det(\mathcal{E}_k^b)_{ij} \geq 0\). Next, we have that

\[
\begin{align*}
&\frac{(b_{i,k}+\epsilon_{i,k})^2}{4} - a_{i,k} f_{i,k} \\
= &\left(2N(\mathcal{E}^b \mathcal{Y}^g)_{ij}\right)^{-2} \left[\left(\bar{q}_k\right)_i^2 + \left(\bar{q}_k\right)_i^{2+i} + \left(\mathcal{Q}_i\right)_i + \left(\mathcal{Q}_i\right)_i^{2+i} \left(\frac{\partial^2 y_i^b + \partial^2 y_i^b}{2}\right)^2\right] - \bar{g}^2 + \\
&\left(\bar{q}_k\right)_i + \left(\bar{q}_k\right)_i^{2+i} \left(\frac{\partial^2 y_i^b + \partial^2 y_i^b}{2}\right)^2 \right) \left(\bar{q}_k\right)_i + \left(\bar{q}_k\right)_i^{2+i} \left(\frac{\partial^2 y_i^b + \partial^2 y_i^b}{2}\right)^2 \right) \right) \\
\leq &\frac{(\bar{q}_k)_{i}^2 + (\bar{q}_k)_{i+i}^2}{4} - \frac{\bar{q}^2}{4} \leq 0,
\end{align*}
\]
which implies that $\det(\mathbf{C}_k^h)^i_i \geq 0$. Next, with a similar procedure, we have that

$$
\begin{align*}
\frac{a_{i,k}(b_{i,k}+\varepsilon_{i,k})}{4} + \frac{a_{i,k}(b_{i,k}+\varepsilon_{i,k})}{4} + f_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 - (b_{i,k}+\varepsilon_{i,k})(b_{i,k}+\varepsilon_{i,k})(h_{i,k}+\varepsilon_{i,k}) \\
-\frac{a_{i,k} f_{i,k} g_{i,k}}{4} & \leq \tilde{g} \left[ -\left\{ - 2N(C^h)(\nabla^2 \mathbf{Y})_{i,j} \right\}^{-1} \left( \left( \mathbf{q}_k \right)_i (\partial_i^2 y_{i,j}^h) + \left( \mathbf{q}_k \right)_i + m \left( \frac{\partial_i^2 y_{i,j}^h + \partial_j^2 y_{i,j}^h}{2} \right) \right) \\
+ \left( \mathbf{q}_k \right)_{i+2m}^2 + \left( \mathbf{q}_k \right)_{i+2m}^2 + \left( \mathbf{q}_k \right)_{i+2m}^2 \right] \leq 0,
\end{align*}
$$

which yields that $\det(\mathbf{C}_k^h)^i_i \geq 0$. Finally, we have that

$$
\begin{align*}
\frac{a_{i,k} f_{i,k}(p_{i,k}+\varepsilon_{i,k})}{4} + \frac{a_{i,k} u_{i,k}(l_{i,k}+\varepsilon_{i,k})}{4} + \frac{a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})}{4} + f_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 \\
+ f_{i,k} u_{i,k}(d_{i,k}+r_{i,k})^2 + a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 - a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})(l_{i,k}+s_{i,k}) \\
- a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})+l_{i,k}+s_{i,k}) (l_{i,k}+s_{i,k}) + \frac{a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})}{4} + f_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 \\
- a_{i,k} u_{i,k}(d_{i,k}+r_{i,k})^2 + a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 - a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})(l_{i,k}+s_{i,k}) \\
- a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})+l_{i,k}+s_{i,k}) (l_{i,k}+s_{i,k}) + \frac{a_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})}{4} + f_{i,k} u_{i,k}(b_{i,k}+\varepsilon_{i,k})^2 \\
= \tilde{g}^2 \left[ -\tilde{g} \left\{ \mathbf{q}_k \right\}_i \left( \partial_i^2 y_{i,j}^h \right) + \left( \mathbf{q}_k \right)_{i+2m} \left( \frac{\partial_i^2 y_{i,j}^h + \partial_j^2 y_{i,j}^h}{2} \right) \right] \\
+ \left( \mathbf{q}_k \right)_{i+2m}^2 + \left( \mathbf{q}_k \right)_{i+2m}^2 + \left( \mathbf{q}_k \right)_{i+2m}^2 \right] \leq 0,
\end{align*}
$$

and then $\det(\mathbf{C}_k^h)^i_i \geq 0$. Therefore, we conclude that $\mathbf{C}_k^h$ is positive definite for $i = 1, \ldots, m$, which implies, by revolving the reordering of the indices, the positive definiteness of matrix $\mathbf{C}_k^h$.

REFERENCES


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